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Variae considerationes circa series hypergeometricas

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VARIAE CONSIDERATIONES
CIRCA
SERIES HYPERGEOMETRICAS.

Auctore
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1.

Proposito hoc producto in infinitum excurrente:
$$\frac{a(a+2b)}{(a+b)(a+b)} \cdot \frac{(a+2b)(a+4b)}{(a+3b)(a+3b)} \cdot \frac{(a+4b)(a+6b)}{(a+5b)(a+5b)} \cdot \frac{(a+6b)(a+8b)}{(a+7b)(a+7b)} \text{ \&c.} = \frac{P}{Q},$$
constat esse
$$P = \int \frac{x^{a+b-1}\partial x}{\sqrt{(1-x^{2b})}} \text{ et } Q = \int \frac{x^{a-1}\partial x}{\sqrt{(1-x^{2b})}},$$
his integralibus ab $x=0$ ad $x=1$ extensis; vbi notetur illius producti membrum indici i respondens esse
$$\frac{[a+(2i-2)b](a+2ib)}{[a+(2i-1)b]\,[a+(2i-1)b]}.$$

§. 2. Iam occasione istius producti consideremus sequens productum indefinitum, in quo factorum numerus sit $=n$, ac ponatur
$$a(a+2b)(a+4b)(a+6b)\ldots\ldots[a+(2n-2)b] = \Delta:n,$$
siquidem hoc productum, ob a & b numeros datos, spectari potest tanquam certa functio ipsius n; ex eius igitur natura perspicuum est fore
$$\Delta:(n+1) = \Delta:n\,.\,(a+2nb),$$
similique modo
$$\Delta:(n+2) = \Delta:(n+1)\,.\,[a+(2n+2)b],$$
et ita porro.

 Hinc

Hinc si i denotet numerum infinite magnum, erit

$$\Delta : i = a(a+2b)(a+4b)(a+6b)\ldots\ldots[a+(2i-2)b]$$

vnde pariter colligitur fore

$$\Delta : (i+1) = \Delta : i\,.\,(a+2ib),$$

$$\Delta : (i+2) = \Delta : i\,(a+2ib)[a+(2i+2)b],$$

$$\Delta : (i+3) = \Delta : i\,(a+2ib)[a+(2i+2)b][a+(2i+4)b]$$

ubi factores insuper accedentes tanquam inter se aequales sp ctari poterunt; quamobrem in genere statui poterit $\Delta : (i +$ $= \Delta : i\,(a+2ib)^n$, vbi cum $(a+2ib)$ sit factor proxi sequens, eodem iure quilibet sequentium sumi potuisset, quo adhuc generalius statuere poterimus

$$\Delta : (i+n) = (\alpha+2ib)^n\,\Delta : i,$$

denotante α numerum quemcunque finitum, quippe qui pr $2ib$ euanescit.

§. 3. In computum nunc ducamus casum producti definiti, quo $n = \frac{1}{2}$, ac vocemus $\Delta : \frac{1}{2} = k$, quem valorem o methodi interpolationum semper vero proxime assignare lic Hinc igitur per superiora erit

$$\Delta : (1+\tfrac{1}{2}) = k(a+b);$$

$$\Delta : (2+\tfrac{1}{2}) = k(a+b)(a+3b),$$

$$\Delta : (3+\tfrac{1}{2}) = k(a+b)(a+3b)(a+5b),$$

vnde in infinitum progrediendo erit

$$\Delta : (i+\tfrac{1}{2}) = k(a+b)(a+3b)(a+5b)\ldots\ldots[a+(2i-1)b]$$

§. 4. Cum igitur supra iam dederimus formulam p $\Delta : (i+n)$, posito nunc $n = \frac{1}{2}$ habebimus quoque

$$\Delta : (i+\tfrac{1}{2}) = \Delta : i\,\sqrt{(\alpha+2ib)},$$

sicque pro eadem formula $\Delta : (i+\frac{1}{2})$ nacti sumus duas diue sas expressiones, ex iisque conficitur ista aequatio:

$$\Delta : i \sqrt{(\alpha + 2ib)} = k(a+b)(a+3b)(a+5b) \ldots . [a+(2i-1)b],$$

atque hinc concludere poterimus, valorem ipsius producti infiniti

$$(a+b)(a+3b)(a+5b) \ldots .. [a+(2i-1)b] = \frac{\Delta : i \sqrt{(\alpha+2ib)}}{k},$$

sicque innotescit relatio inter hoc productum et id quod supra per $\Delta : i$ expressimus. Hic autem probe notandum est factores huius producti eos ipsos esse, qui denominatorem producti initio propositi constituunt, quamobrem tam numeratorem illius producti quam denominatorem per valores modo inuentos $\Delta : i$ et $\frac{\Delta : i \sqrt{(\alpha + 2ib)}}{k}$ exprimere poterimus.

§. 5. Numerator autem producti propositi in infinitum expansus ita repraesentari potest:

$$a(a+2b)^2 (a+4b)^2 \ldots \ldots [a+(2i-2)b]^2 (a+2ib),$$

vbi factores primus et vltimus sunt solitarii, reliqui vero omnes quadrati. Cum igitur sit

$$(\Delta : i)^2 = (a)^2 (a+2b)^2 (a+4b)^2 (a+6b)^2 \ldots .. [a+(2i-2)b]^2,$$

euidens est illum numeratorem esse $\frac{(\Delta : i)^2}{a}(a+2ib)$. Pro denominatore autem per se manifestum est, eum esse aequalem quadrato producti alterius $(a+b)(a+3b)$ &c., cuius valor cum repertus sit $\frac{\Delta : i \sqrt{(\alpha + 2ib)}}{k}$, denominator erit $\frac{(\Delta : i)^2 (\alpha + 2ib)}{kk}$, quamobrem his valoribus substitutis pro fractione supra exposita $\frac{P}{Q}$ affecuti sumus hanc aequationem:

$$\frac{P}{Q} = \frac{\frac{(\Delta : i)^2 (a + 2ib)}{a}}{\frac{(\Delta : i)^2 (\alpha + 2ib)}{kk}} = \frac{kk(a+2ib)}{a(\alpha+2ib)} = \frac{kk}{a},$$

Ex hac aequatione igitur statim innotescit verus valor formulae interpolatae $k = \Delta : \frac{1}{2}$, quandoquidem erit $\Delta : \frac{1}{2} = \sqrt{\frac{aP}{Q}}$, atque hinc porro sequentes:

$$\Delta:(1+\tfrac{1}{2}) = (a+b)\sqrt{\tfrac{aP}{Q}},$$
$$\Delta:(2+\tfrac{1}{2}) = (a+b)(a+3b)\sqrt{\tfrac{aP}{Q}},$$
$$\Delta:(3+\tfrac{1}{2}) = (a+b)(a+3b)(a+5b)\sqrt{\tfrac{aP}{Q}}, \text{ etc.}$$

haecque interpolatio eo magis eſt notatu digna, quod ſine approximatione ſtatim verum valorem horum terminorum interpolatorum ſuppeditat.

§. 6. Quod ſi inſuper iſtud productum infinitum, in quo vtrique factores coniunguntur, contemplemur, ac ſtatuamus

$$a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(i-1)b] = \Gamma:i, \text{ erit}$$
$$\Gamma:2i = a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(2i-1)b],$$

quod manifeſto eſt productum ex binis ſuperioribus, ita vt ſit $\Gamma:2i = \frac{(\Delta:i)^2\sqrt{(a+2ib)}}{k}$; vnde ſi forma $\Gamma:2i$ vti voluerimus, valores amborum praecedentium ex eo aſſignare poterimus, cum ſit $\Delta:i = \sqrt{\frac{k.\Gamma:2i}{\sqrt{(a+2ib)}}}$, qui eſt ipſe valor prioris producti

$$a(a+2b)(a+4b)(a+6b), \text{ etc.}$$

alterius vero producti

$$(a+b)(a+3b)(a+5b) \text{ etc. valor erit } \sqrt{\frac{\Gamma:2i\sqrt{(a+2ib)}}{k}}.$$

§. 7. Hactenus igitur tria producta in infinitum excurrentia atque inter ſe affinia ſumus contemplati, quae, quoniam ea accuratius ſumus perſcrutaturi, hic denuo ob oculos exponamus

I. $a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(i-1)b] = \Gamma:i,$

II. $a(a+2b)(a+4b)(a+6b)\ldots\ldots[a+(2i-2)b] = \Delta:i,$

III. $(a+b)(a+3b)(a+5b)\ldots\ldots[a+(2i-1)b] = \Theta:i,$

atque iam inuenimus eſſe $\Theta:i = \frac{\Delta:i\sqrt{(a+2ib)}}{k}$; tum vero tam $\Delta:i$ quam $\Theta:i$ ſequenti modo per functionem $\Gamma:2i$ expreſſimus:

$\Delta:i$

$$\Delta : i = \sqrt{\frac{k \cdot \Gamma : 2i}{\sqrt{(a+2ib)}}} \text{ et } \Theta : i = \sqrt{\frac{\Gamma : 2i\sqrt{(a+2ib)}}{k}},$$

quandoquidem manifestum est esse $\Gamma : 2i = \Delta : i \cdot \Theta : i$; vbi meminisse oportet esse $k = \Delta : \frac{1}{2}$, quod scilicet ex forma secunda definiri debet, considerando seriem

$$a,\ a(a+2b),\ a(a+2b)(a+4b),\ a(a+2b)(a+4b)(a+6b), \text{ etc.}$$

cuius terminum indici $\frac{1}{2}$ respondentem designauimus hac litterâ k.

§. 8. Iam ad istas formas accommodemus methodum generalem summandi omnis generis progressiones per terminum earum generalem, quae ita se habet, vt proposita serie quacunque A, B, C, D, E, etc. cuius terminus indici indefinito x respondens sit $= X$, eius summa

$$A + B + C + D + \ldots\ldots\ldots + X,$$

quam vocemus $= S$, sit

$$S = \int X \partial x + \tfrac{1}{2} X + \frac{1}{1.2.3} \cdot \frac{1}{2} \frac{\partial X}{\partial x} - \frac{1}{1\ldots5} \cdot \frac{1}{6} \frac{\partial^3 X}{\partial x^3} + \frac{1}{1\ldots7} \cdot \frac{1}{6} \frac{\partial^5 X}{\partial x^5} - \text{etc.}$$

vbi fractiones $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$, etc. sunt numeri Bernoulliani.

Euolutio formae primae.

$$a(a+b)(a+2b)(a+3b)\ldots\ldots[a+(i-1)b] = \Gamma : i.$$

§. 9. Cum numerus factorum hic consideretur vt infinitus, quo methodum summandi ad eam applicare valeamus, consideremus eandem formam numero terminorum finito $= x$ constantem, ac statuamus simili modo

$$a(a+b)(a+2b)(a+3b)\ldots\ldots\ldots[a+(x-1)b] = \Gamma : x.$$

Nunc vero vt loco huius producti seriem summandam nanciscamur, sumamus logarithmos, eritque

$$l\Gamma : x = la + l(a+b) + l(a+2b) + l(a+3b) \ldots\ldots l[a+(x-1)b],$$

cuius ergo summa cum fuerit explorata, dabit logarithmum formulae

mulae $\Gamma : x$, ideoque ipsam formulam $\Gamma : x$, in qua si deinceps statuatur $x = i$, obtinebitur formula $\Gamma : i$, quem valorem in superioribus potissimum spectauimus. Hinc igitur, comparatione cum serie generalissima instituta, erit $X = l\,[a + (x - 1)\,b]$, siue erit $X = l\,(a - b + b\,x)$, atque ipsa summa $S = l\,\Gamma : x$, vnde colligitur $\int X\,\partial x = \int \partial x\, l\,(a - b + b\,x)$.

§. 10. Cum igitur sit $\int \partial z\, l\, z = z\, l\, z - z$, atque

$$\int \partial y\, l\,(a + y) = (a + y)\, l\,(a + y) - (a + y),$$

nunc loco y scribendo $b\,x$ erit

$$\int b\,\partial x\, l\,(a + b\,x) = (a + b\,x)\, l\,(a + b\,x) - a - b\,x,$$

ideoque

$$\int \partial x\, l\,(a + b\,x) = \frac{(a + b\,x)}{b}\, l\,(a + b\,x) - \frac{a}{b} - x,$$

vnde colligitur pro nostro casu fore:

$$\int X\,\partial x = \frac{(a - b + b\,x)}{b}\, l\,(a - b + b\,x) - \frac{a}{b} + 1 - x,$$

vbi in vltima parte membrum constans $\frac{a}{b} - 1$ omittere licet, quia expressio constantem quantitatem indefinitam per se postulat, quam deinceps ex indole seriei definiri oportet. Deinde vero erit $\frac{\partial X}{\partial x} = \frac{b}{a - b + b\,x}$; tum vero porro

$$\frac{\partial^3 X}{\partial x^3} = \frac{2\,b^3}{(a - b + b\,x)^3};\quad \frac{\partial^5 X}{\partial x^5} = \frac{2.\,3.\,4\,b^5}{(a - b + b\,x)^5};\ \text{etc.}$$

quibus valoribus in vsum vocatis erit

$$l\,\Gamma : x = A + \left(\frac{a}{b} - \frac{1}{2} + x\right) l\,(a - b + b\,x) - x + \frac{1}{1.\,2.\,3}\cdot\frac{1}{2}\cdot\frac{b}{a - b + b\,x}$$
$$- \frac{1}{3.\,4.\,5}\cdot\frac{1}{6}\cdot\frac{b^3}{(a - b + b\,x)^3} + \frac{1}{5.\,6.\,7}\cdot\frac{1}{6}\cdot\frac{b^5}{(a - b + b\,x)^5}$$
$$- \frac{1}{7.\,8.\,9}\cdot\frac{3}{10}\cdot\frac{b^7}{(a - b + b\,x)^7} + \frac{1}{9.\,10.\,11}\cdot\frac{5}{6}\cdot\frac{b^9}{(a - b + b\,x)^9} - \text{etc.}$$

vbi littera A constantem denotat ex indole seriei definiendam.

§. 11. Constans autem ista A ex casu quo summa seriei est cognita determinari debet, quod ergo fieri posset casu $x = 0$, quippe summa etiam nihilo aequalis prodire debet; foret igitur hinc

$$A = \left(\frac{a}{b} - \frac{1}{2}\right) l\,(a - b) + \frac{1}{1.2.3} \cdot \frac{1}{2} \cdot \frac{b}{a - b} - \frac{1}{3.4.5} \cdot \frac{1}{6}\, \frac{b^3}{(a - b)^3} + \frac{1}{5.6.7} \cdot \frac{1}{6}\, \frac{b^5}{(a - b)^5} - \text{etc.}$$

Quoniam autem haec series parum conuergit, atque adeo casu $b = a$ omnes termini fierent infiniti, hinc nihil plane lucrari licet. Sin autem vellemus sumere $x = 1$, ista summa prodire deberet $= l\,a$; vnde pariter vix quicquam pro instituto nostro concludere liceret, quia semper ad seriem infinitam perveniretur, cuius summam demum explorare oporteret, in quo quidem negotio forsitan ea, quae olim de seriebus numeros Bernoullianos inuoluentibus sum commentatus, aliquem vsum praestare possent, cui autem labori nunc immorari non vacat.

§. 12. Quia enim in praesenti instituto potissimum ad valorem $\Gamma : i$ respicimus, sufficiet statim loco x numerum infinitum statui. Sit igitur $x = i$, denotante i numerum infinite magnum, et aequatio nostra hanc induet formam:

$$l\,\Gamma : i = A + \left(\frac{a}{b} - \frac{1}{2} + i\right) l\,(a - b + b\,i) - i,$$

vnde constans ista A sponte determinatur, quam idcirco quasi iam cognitam spectabimus. Hinc ergo ad numeros regrediendo, vbi quidem loco A scriptum intelligamus $l\,A$, perueniemus ad hanc expressionem:

$$\Gamma : i = A\,(a - b + b\,i)^{\frac{a}{b} - \frac{1}{2} + i} \cdot e^{-i}.$$

Hic quidem conueniet potestatem exponentis i seorsim repraesentare hoc modo:

$$\Gamma : i = A\,(a - b + b\,i)^{\frac{a}{b} - \frac{1}{2}} (a - b + b\,i)^i \cdot e^{-i}.$$

Euolutio binarum reliquarum formularum.

§. 13. Secunda forma a prima in eo tantum discre-
pat, quod loco b hic scribi oportet $2b$, vnde noua euolutio
ne carere possumus; at vero loco constantis A hic scribam
B, quandoquidem nondum constat, quemadmodum littera b
constantem A ingrediatur. Hoc modo igitur statim habebim

$$\Delta : i = B\,(a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}}\,(a - 2b + 2bi)^{i}\,e^{-i}$$

Simili modo euidens est, ex hac secunda forma oriri tertia
si modo loco a scribatur $a + b$, vnde loco B constantem
introducendo, habebimus

$$\Theta : i = C\,(a - b + 2bi)^{\frac{a}{2b}}\,(a - b + 2bi)^{i}\,e^{-i}.$$

Vbi notetur litteram e hic positam esse pro numero cuius
garithmus hyperbolicus $= 1$.

Conclusiones hinc oriundae.

§. 14. Videamus nunc quomodo istae nouae deter
nationes se respectu relationum supra inuentarum sint hab
rae; quare cum ex his nouis valoribus sit

$$\Gamma : 2i = A\,(a - b + 2bi)^{\frac{a}{b} - \frac{1}{2}}\,(a - b + 2bi)^{2i}\,e^{-}$$

quia inuenimus:

$$\Gamma : 2i = \Delta : i \times \Theta : i,$$

si vbique hic valores modo inuentos substituamus, habebi
pro ista aequatione primo productum:

$$\Delta : i \,.\, \Theta : i = B\,C\,(a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}}\,(a - b + 2bi$$

$$(a - 2b + 2bi)^{i}\,(a - b + 2bi)^{i}\,e^{-2i},$$

quod cum esse debeat illi valori $\Gamma : 2i$ aequale, si per fa
res quos habeat communes vtrinque diuidamus, prodibit
aequatio:

$$A\,(a-b+2\,b\,i)^{\frac{a}{2b}-\frac{1}{2}}\,(a-b+2\,b\,i)^{i}$$
$$=B\,C\,(a-2\,b+2\,b\,i)^{\frac{a}{2b}-\frac{1}{2}}\,(a-2\,b+2\,b\,i)^{i}.$$

§. 15. Diuidamus hanc aequationem vtrinque per $(a-2\,b+2\,b\,i)^{i}$, et cum fit

$$\frac{a-b+2\,b\,i}{a-2\,b+2\,b\,i}=1+\frac{b}{a-2\,b+2\,b\,i}=1+\frac{1}{2\,i},$$

ob i numerum infinitum, per refolutionem ordinariam erit $(1+\frac{1}{2\,i})^{i}=e^{\frac{1}{2}}$, vnde aequatio noftra reducitur ad hanc formam:

$$A\,(a-b+2\,b\,i)^{\frac{a}{2b}-\frac{1}{2}}\times e^{\frac{1}{2}}=B\,C\,(a-2\,b+2\,b\,i)^{\frac{a}{2b}-\frac{1}{2}},$$

vbi vltimi factores quoque fe tollunt, cum fit

$$\left(\frac{a-b+2\,b\,i}{a-2\,b+2\,b\,i}\right)^{\frac{a}{2b}-\frac{1}{2}}=\left(1+\frac{1}{2\,i}\right)^{\frac{a}{2b}-\frac{1}{2}}=1,$$

ita vt peruentum fit ad hanc fimplicem aequalitatem:

$$A\,e^{\frac{1}{2}}=B\,C.$$

§. 16. Deinde cum fupra inuenerimus effe

$$\Theta:i=\frac{\Delta\,i\,\surd(\alpha+2\,i\,b)}{k},\ \text{fiue}\ \frac{\Theta:i}{\Delta:i}=\frac{\surd(\alpha+2\,i\,b)}{k},$$

diuidamus valorem pro $\Theta:i$ inuentum per $\Delta:i$, ac reperiemus:

$$\frac{\Theta:i}{\Delta:i}=\frac{C}{B}\surd(a-2b+2\,b\,i)\left(\frac{a-b+2\,b\,i}{a-2\,b+2\,b\,i}\right)^{i}=\frac{C}{B}\surd\,e\,(a-2b+2\,b\,i).$$

Erit ergo

$$\frac{\surd(\alpha+2\,i\,b)}{k}=\frac{C}{B}\surd\,e\,(a-2\,b+2\,b\,i),\ \text{fiue}$$

$$\frac{1}{k}=\frac{C}{B}\surd\frac{e\,(a-2\,b+2\,b\,i)}{\alpha+2\,i\,b}=\frac{C}{B}\surd\,e,$$

fiue erit $B=C\,k\,\surd\,e$.

§. 17. Nacti ergo fumus eiusmodi binas relationes inter ternas illas conftantes A, B, C, vt fi earum vnica effet cognita, ex ea binae reliquae definiri poffent. Cum enim fit $A = \frac{BC}{\sqrt{e}}$ et $B = Ck\sqrt{e}$, fi conftantem A fpectemus vt iam cognitam, binae reliquae fequenti modo determinabuntur: Cum fit $B = Ck\sqrt{e}$, hic valor in priore aequatione fubftitutus dat $A = CCk$, vnde elicitur $C = \sqrt{\frac{A}{k}}$, hincque porro $B = \sqrt{kA}e$. Interim tamen hinc non patet, quomodo conftans A abfolute determinari queat; ideoque recurrendum erit ad ipfam illam fummationem feriei logarithmicae, quam littera A fupra indicauimus; vbi autem loco A fcribendum erit $l\,A$. Atque hinc tantum fumus lucrati, vt fi binae reliquae formae fimili modo euoluantur per feries logarithmicas, conftantes ibi adhibendae fcilicet $l\,B$ et $l\,C$ fimul innotefcant.

§. 18. Supereft vt adhuc pauca addamus de valore litterae k, quam per interpolationem inueniri debere iam fupra monuimus. Interim tamen haec littera etiam ex ipfa comparatione formularum $\Delta : i$ et $\Theta : i$ abfolute per certas quadraturas determinari poteft. Cum enim fit

$$k = \frac{\Delta : i}{\Theta : i}\sqrt{(\alpha + 2ib)}, \text{ ideoque}$$

$$kk = \frac{(\Delta : i)^2(\alpha + 2ib)}{(\Theta : i)^2},$$

fi loco $\Delta : i$ et $\Theta : i$ ipfa producta infinita fubftituamus, et quoniam vtrumque ex i factoribus conftat, hic autem in numeratore vnus factor $\alpha + 2ib$ infuper accedit, primum numeratoris factorem feorfim exprimamus, hoc pacto perueniemus ad fequens productum determinatum:

$$kk = a \cdot \frac{a(a+2b)(a+2b)(a+4b)(a+4b)(a+6b)}{(a+b)(a+b)(a+3b)(a+3b)(a+5b)(a+5b)} \text{ etc.}$$

§. 19

§. 19. Vt autem huius producti infiniti verum valorem eruamus, recordandum eſt, ſi litterae P et Q denotent ſequentes formulas integrales:

$$P = \int \frac{x^{p-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}} \text{ et } Q = \int \frac{x^{q-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

quae integralia ab $x = 0$ ad $x = 1$, extendi ſunt intelligenda, tum per productum infinitum fore:

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

quod productum facile ad noſtram formam reducitur, ſumendo $q = a$, $p = a+b$, $m = b$, $n = 2b$, ita vt pro noſtro caſu fiat

$$P = \int \frac{x^{a+b-1}\partial x}{\sqrt{(1-x^{2b})}} \text{ et } Q = \int \frac{x^{a-1}\partial x}{\sqrt{(1-x^{2b})}};$$

tum vero erit $kk = \frac{aP}{Q}$, ideoque $k = \sqrt{\frac{aP}{Q}}$, ſicque eundem valorem k alio modo elicuimus, quem iam ſupra attulimus.

§. 20. Quemadmodum autem eſt $k = \Delta : \frac{1}{2}$, ſimili modo pro binis reliquis formis poterimus aſſignare valores $\Gamma : \frac{1}{2}$ et $\Theta : \frac{1}{2}$. Cum enim forma Γ oriatur ex forma Δ, ſi in hac loco b ſcribatur $\frac{1}{2}b$, forma autem Θ oriatur ex Δ, ſi loco a ſcribatur $a+b$, his obſeruatis erit

$$\Gamma : \tfrac{1}{2} = \sqrt{a\frac{\int x^{a+\frac{1}{2}b-1}\partial x : \sqrt{(1-x^b)}}{\int x^{a-1}\partial x : \sqrt{(1-x^b)}}} \text{ et}$$

$$\Theta : \tfrac{1}{2} = \sqrt{(a+b)\frac{\int x^{a+2b-1}\partial x : \sqrt{(1-x^{2b})}}{\int x^{a+b-1}\partial x : \sqrt{(1-x^{2b})}}}.$$

Facile autem intelligitur, valorem $\Theta : \frac{1}{2}$ aeque in noſtros calculos introduci potuiſſe ac $\Delta : \frac{1}{2} = k$, cum ſit $\Delta : \frac{1}{2} . \Theta . \frac{1}{2} = a$.

 Ductis

Ductis enim in ſe inuicem illis valoribus integralibus prodit

$$\Delta : \tfrac{1}{2} . \Theta : \tfrac{1}{2} = \sqrt{\frac{a(a+b)\int x^{a+2b-1}\partial x : \sqrt{(1-x^{2b})}}{\int x^{a-1}\partial x : (1-x^{2b})}}$$

ex notiſſima autem reductione talium integralium conſtat eſſe

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt{(1-x^{2b})}} = \frac{a}{a+b}\int \frac{x^{a-1}\partial x}{\sqrt{(1-x^{2b})}},$$

pro terminis ſcilicet integrationis $x = 0$ et $x = 1$, ſicque perſpicuum eſt fore $\Delta : \tfrac{1}{2} . \Theta : \tfrac{1}{2} = a$. Quemadmodum autem valor $\Gamma : \tfrac{1}{2}$ ad binos reliquos referatur, nullo modo definiri poteſt.